

Pseudo-Anosov extensions and degree one maps between hyperbolic surface bundles

BY MICHEL BOILEAU

*Laboratoire Émile Picard,
Université Paul Sabatier, TOULOUSE Cedex 4, France
e-mail: boileau@picard.ups-tlse.fr*

YI NI

*Department of Mathematics,
Princeton University, NJ 08544, U. S. A.
e-mail: yni@math.princeton.edu*

AND SHICHENG WANG

*LMAM, Department of Mathematics,
Peking University, Beijing 100871 China
e-mail: wangsc@math.pku.edu.cn*

Abstract

Let F', F be any two closed orientable surfaces of genus $g' > g \geq 1$, and $f : F \rightarrow F$ be any pseudo-Anosov map. Then we can “extend” f to be a pseudo-Anosov map $f' : F' \rightarrow F'$ so that there is a fiber preserving degree one map $M(F', f') \rightarrow M(F, f)$ between the hyperbolic surface bundles. Moreover the extension f' can be chosen so that the surface bundles $M(F', f')$ and $M(F, f)$ have the same first Betti numbers.

1. Introduction

All surfaces are oriented and all automorphisms on surfaces are orientation preserving.

Let F be an oriented closed surface of genus $g \geq 1$, and $f : F \rightarrow F$ be an automorphism. We denote the surface bundle with fiber F and monodromy f by $M(F, f)$.

Definition 1.1. Suppose G is a compact surface of genus $g \geq 1$. A circle c on G is *essential* if c is neither contractible nor boundary parallel. An automorphism f of G is *pseudo-Anosov* if $f^n(c)$ is not isotopic to c for any essential circle $c \subset G$ and any integer n . (Note in the case G is a torus, the term “pseudo-Anosov” we define here is usually known as “Anosov”.)

Remark 1.2. Our definition of pseudo-Anosov maps is slightly different from the more standard definition in the literature. Pseudo-Anosov maps in our sense should be considered as “maps isotopic to a pseudo-Anosov map” in the standard sense.

Profound theories of Nielsen-Thurston and of Thurston in 2- and 3-dimensional topology tell us that pseudo-Anosov is the most important class of surface automorphisms, and when $\chi(F) < 0$, $M(F, f)$ is a hyperbolic 3-manifold if and only if f is pseudo-Anosov.

THEOREM 1.3. *Let F_s, F_t be closed orientable surfaces of genus g_s, g_t respectively, $g_s > g_t \geq 1$, and $f_t : F_t \rightarrow F_t$ be a pseudo-Anosov map. Then*

(1) *There exists a hyperbolic 3-manifold $M(F_s, f_s)$, such that there is a fiber preserving degree one map $P : M(F_s, f_s) \rightarrow M(F_t, f_t)$. (Here the subscript s means “source”, and the subscript t means “target”).*

(2) *Moreover the f_s in (1) can be chosen so that $M(F_s, f_s)$ and $M(F_t, f_t)$ have the same first Betti numbers.*

Motivation for Theorem 1.3 is from [1], where the following facts were proved:

(1) For each 3-manifold M , there is a degree one map $f : M(F_s, f_s) \rightarrow M$ such that $M(F_s, f_s)$ is hyperbolic and $\beta_1(M(F_s, f_s)) = \beta_1(M) + 1$.

(2) If there is a degree one map $f : M(F_s, f_s) \rightarrow M$ with $\beta_1(M(F_s, f_s)) = \beta_1(M)$ and M is irreducible, then M is a surface bundle and f can be homotoped to a fiber preserving one.

It is natural to wonder how to find fiber preserving degree one maps between non-homeomorphic hyperbolic surface bundles (of the same first Betti numbers). In Section 2, we will prove Theorem 1.3 (1). In Section 3, by modifying the proof in Section 2, we will prove Theorem 1.3 (2).

The proof of Theorem 1.3 relies on an extension process from the pseudo-Anosov map f_t on F_t to a pseudo-Anosov map f_s on F_s , which is delicate and somewhat complicated.

We will outline this process, i.e., for given $M(F_t, f_t)$ and F_s , how to find f_s . In this outline we assume that $g_t \geq 2$. This process in Section 2 is divided into three steps.

Step 1. Fix a disk $D \subset F_t$ and let $V = F_t - \text{int}(D)$. We can assume that $f_t|_D = \text{id}|_D$ up to isotopy. Then as a restriction of a pseudo-Anosov map, $f_t|_V : V \rightarrow V$ is a pseudo-Anosov map (Lemma 2.4).

Step 2. We will construct two embeddings $e_0, e_1 : V \rightarrow F_s$ such that (1) $e_0(\partial V)$ and $e_1(\partial V)$ are not homotopic in F_s , (2) two pinches $p_0, p_1 : F_s \rightarrow F_t$ (see Definition 2.1) defined by $p_j \circ e_j = \text{id}_V : V \rightarrow V$ are homotopic (Lemma 2.6).

Step 3. The two embeddings e_1 and e_2 in step 2 also provided a homeomorphism $\bar{f}_t := e_1 \circ f_t \circ e_0^{-1} : e_0(V) \rightarrow e_1(V)$. With properties of e_1 and e_2 described in Step 2, we will be able to extend \bar{f}_t to a pseudo-Anosov map $f_s : F_s \rightarrow F_s$ (Proposition 2.7).

Then clearly $p_1 \circ f_s = f_t \circ p_0$, hence there exists a fiber preserving degree one map $P : M(F_s, f_s) \rightarrow M(F_t, f_t)$ (Lemma 2.3). This finishes the proof of Theorem 1.3 (1).

Now we give more detailed outline of the extension process in Step 3, on which the proof of Theorem 1.3 (2) is based.

Let $\bar{f}_s : F_s \rightarrow F_s$ be any extension of $\bar{f}_t : e_0(V) \rightarrow e_1(V)$ (Lemma 2.4). Let $W_1 = F_s - \text{int}e_1(V)$ and $h : W_1 \rightarrow W_1$ be any pseudo-Anosov map. Let \mathcal{A}_1 be any maximal independent set of disjoint circles on W_1 (see Definition 2.9), let $\tau(\mathcal{A}_1)$ be a composition of Dehn twists along all components in \mathcal{A}_1 . Then $f_s = \tau^l(\mathcal{A}_1) \circ h^k \circ \bar{f}_s$ is pseudo-Anosov for large integers k and l (Lemmas 2.8, 2.10, 2.11, 2.12, 2.14).

In Section 3, we choose \bar{f}_s , h and \mathcal{A}_1 carefully so that Theorem 1.3 (2) is proved (Lemmas 3.1, 3.2, 3.3).

We end the introduction by a comment on a related work [9]. The main result in [9] is that for an orientable closed surface F with $\chi(F) < 0$ and two non-isotopic circles c and c' on F , if $g(c) = c'$ for some automorphism g on F , then $f(c) = c'$ for some pseudo-Anosov map f on F . Some arguments in proving Lemmas 2.10 and 2.12 were influenced

by that in [9]. Indeed [9] is produced in a rather earlier stage of understanding the present project.

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2. Homotopic pinches and pseudo-Anosov extensions

Definition 2.1. Let D be a fixed disc in F_t and $V = F_t - \text{int}(D)$. A degree one map $p : F_s \rightarrow F_t$ is a *pinch* if $p| : p^{-1}(V) \rightarrow V$ is a homeomorphism.

It has been known since Nielsen and Kneser that every degree one map between surfaces is homotopic to a pinch, see [2] for a reference.

Notation 2.2. In the rest of this paper, $r = s, t$ and $j = 0, 1$.

Recall that $M(F_r, f_r) = F_r \times [0, 1]/f'_r$, where $f'_r : F_r \times \{0\} \rightarrow F_r \times \{1\}$ is given by $f'_r(x, 0) = (f_r(x), 1)$. Let $q_r : F_r \times [0, 1] \rightarrow F_r$ be the projection defined by $q_r(x, u) = x$, and $e_{r,j} : F_r \rightarrow F_r \times \{j\} \subset F_r \times [0, 1]$ be the homeomorphism given by $e_{r,j}(x) = (x, j)$. Let $o_r : F_r \times [0, 1] \rightarrow M(F_r, f_r)$ be the quotient map and $F'_r = o_r(F_r \times 0) = o_r(F_r \times 1)$.

Then

$$q_r \circ f'_r \circ e_{r,0} = f_r, \quad e_{r,j} \circ q_r = \text{id}_{F_r \times \{j\}}. \quad (*)$$

LEMMA 2.3. *There exists a fiber preserving degree one map $P : M(F_s, f_s) \rightarrow M(F_t, f_t)$ if and only if there are homotopic pinches $p_0, p_1 : F_s \rightarrow F_t$ such that $p_1 \circ f_s = f_t \circ p_0$.*

Proof. Suppose first that $P : M(F_s, f_s) \rightarrow M(F_t, f_t)$ is a fiber preserving degree 1 map. Up to homotopy we may assume that $P^{-1}(F'_t) = F'_s$ and $P| : F'_s \rightarrow F'_t$ is a pinch. Moreover we may assume that the induced degree one map on S^1 is orientation preserving. Then by cutting $M(F_r, f_r)$ along F'_r , P provides a proper degree one map

$$\bar{P} : (F_s \times [0, 1], F_s \times \{0\}, F_s \times \{1\}) \rightarrow (F_t \times [0, 1], F_t \times \{0\}, F_t \times \{1\})$$

with the property $\bar{P}|_{F_s \times \{1\}} \circ f'_s = f'_t \circ \bar{P}|_{F_s \times \{0\}}$.

Let $p_j = q_t \circ \bar{P}|_{F_s \times \{j\}} \circ e_{s,j} : F_s \rightarrow F_t$. Then p_j is a pinch and $q_t \circ \bar{P} : F_s \times [0, 1] \rightarrow F_t$ is a homotopy from p_0 to p_1 . Moreover $\bar{P}|_{F_s \times \{1\}} \circ f'_s = f'_t \circ \bar{P}|_{F_s \times \{0\}}$ and $(*)$ imply that

$$\begin{aligned} p_1 \circ f_s &= q_t \circ P|_{F_s \times \{1\}} \circ e_{s,1} \circ q_s \circ f'_s \circ e_{s,0} \\ &= q_t \circ \bar{P}|_{F_s \times \{1\}} \circ f'_s \circ e_{s,0} \\ &= q_t \circ f'_t \circ \bar{P}|_{F_s \times \{0\}} \circ e_{s,0} \\ &= q_t \circ f'_t \circ e_{t,0} \circ q_t \circ \bar{P}|_{F_s \times \{0\}} \circ e_{s,0} \\ &= f_t \circ p_0. \end{aligned}$$

Suppose then there are two homotopic pinches $p_0, p_1 : F_s \rightarrow F_t$ such that $p_1 \circ f_s = f_t \circ p_0$. Let $P' : F_s \times [0, 1] \rightarrow F_t$ be a homotopy from p_0 to p_1 . Then P' provides a proper degree one map

$$\bar{P} : (F_s \times [0, 1], F_s \times \{0\}, F_s \times \{1\}) \rightarrow (F_t \times [0, 1], F_t \times \{0\}, F_t \times \{1\})$$

defined by $\bar{P}(x, u) = (P'(x, u), u)$. Clearly \bar{P} is fiber preserving and $p_j = q_t \circ \bar{P}|_{F_s \times \{j\}} \circ e_{s,j}$. Then $p_1 \circ f_s = f_t \circ p_0$ and $(*)$ implies that

$$q_t \circ P|_{F_s \times \{1\}} \circ e_{s,1} \circ q_s \circ f'_s \circ e_{s,0} = q_t \circ f'_t \circ e_{t,0} \circ q_t \circ \bar{P}|_{F_s \times \{0\}} \circ e_{s,0},$$

hence

$$q_t \circ P|_{F_s \times \{1\}} \circ f'_s \circ e_{s,0} = q_t \circ f'_t \circ \bar{P}|_{F_s \times \{0\}} \circ e_{s,0}.$$

Since $q_t|_{F_t \times \{1\}}$ and $e_{s,0}$ are invertible, we have

$$P|_{F_s \times \{1\}} \circ f'_s = f'_t \circ \bar{P}|_{F_s \times \{0\}}.$$

Hence \bar{P} is able to induce a fiber preserving degree one map $P : M(F_s, f_s) \rightarrow M(F_t, f_t)$.

By Lemma 2.3, to prove Theorem 1.3 (1), we need only to find two homotopic pinches $p_0, p_1 : F_s \rightarrow F_t$ and a pseudo-Anosov map $f_s : F_s \rightarrow F_s$ such that $p_1 \circ f_s = f_t \circ p_0$.

For $D \subset F_t$ and $V = F_t - \text{int}(D)$ given in Definition 2.1, we can assume that $f_t|_D = \text{id}$ up to isotopy.

LEMMA 2.4. *If $f_t : F_t \rightarrow F_t$ is a pseudo-Anosov map, then $f_t|_V : V \rightarrow V$ is also a pseudo-Anosov map.*

Proof. Suppose there is a non-contractible circle c on V such that $f_t|_V^n(c) \sim c$ on V for some $n > 0$, then $f_t^n(c) \sim c$ on F_t . Since f_t is pseudo-Anosov, c is contractible on F_t . Hence c bounds a disc D^* in F_t and $D \subset D^*$. It follows that $c = \partial D^*$ is parallel to $\partial D = \partial V$. Hence $f_t|_V : V \rightarrow V$ is pseudo-Anosov by definition.

Let $p_0, p_1 : F_s \rightarrow F_t$ be two pinches. Then the pull-back of V into F_s provides embeddings $e_j : V \hookrightarrow F_s$. Let $V_j = e_j(V)$, $W_j = F_s - \text{int}(V_j)$, ($j = 0, 1$). The following lemma is clear

LEMMA 2.5.

$$\bar{f}_t := e_1 \circ f_t \circ e_0^{-1} : V_0 \rightarrow V_1$$

is a homeomorphism. Moreover, \bar{f}_t can be extended to a homeomorphism $f_s : F_s \rightarrow F_s$, such that $f_t \circ p_0 = p_1 \circ f_s$.

A necessary condition to guarantee the extension f_s in Lemma 2.5 to be pseudo-Anosov is that $e_0(\partial D)$ is **not** homotopic to $e_1(\partial D)$.

Now with Lemma 2.3 and Lemma 2.5, Theorem 1.3 (1) follows from the following Lemma 2.6 and Proposition 2.7.

LEMMA 2.6. *With the notation above, there exist two pinches $p_0, p_1 : F_s \rightarrow F_t$ such that*

- (i) p_0 and p_1 are homotopic;
- (ii) $e_0(\partial D)$ is not homotopic to $e_1(\partial D)$.

Proof. We will find two essential circles $\gamma_0, \gamma_1 \subset F_s$ such that

- (1) γ_0 is not homotopic to γ_1 ,
- (2) γ_j separates F_s into 1-punctured surfaces V_j and W_j , where V_0, V_1 have genus g_t .

Then we define the pinch $p_j : F_s \rightarrow F_t$ such that W_j is the pinched part.

Case 1. $g_t \geq 2$. W_j, V_j , are shown in Figure 1.

Let $p_j : F_s \rightarrow F_t$ be a pinch which sends W_j to $D_j \subset F_t$ such that the restrictions

$p_0|, p_1| : F_s \setminus (W_0 \cup W_1) \rightarrow F_t \setminus (D_0 \cup D_1)$ are identical homeomorphisms. This requirement can be reached if we consider $p_j : F_s \rightarrow F_t$ as a quotient map which is the identity on $F_s \setminus W_j$ and pinches W_j to D_j .

Note $W_0 \cup W_1$ is a compact surface with two boundary components and $D_0 \cup D_1$ is annulus. Moreover the restrictions $p_0|, p_1| : W_0 \cup W_1 \rightarrow D_0 \cup D_1$ are degree one maps which are identity on the boundary, it follows from classical argument that $p_0|, p_1| : W_0 \cup W_1 \rightarrow D_0 \cup D_1$ are homotopic relative to the boundary, and finally $p_0, p_1 : F_s \rightarrow F_t$ are homotopic.

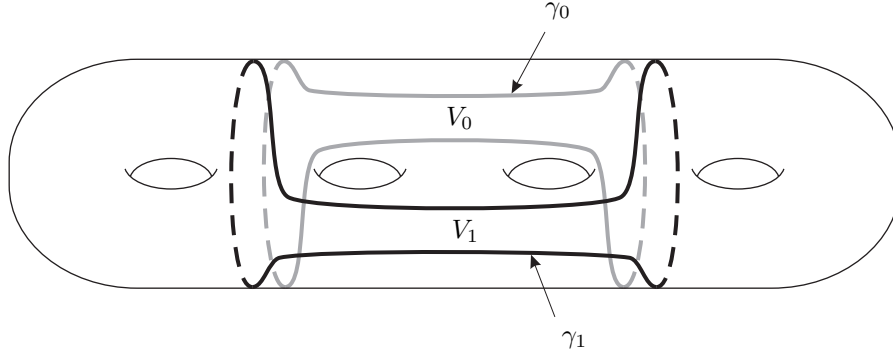


Figure 1

Case 2. $g_t = 1$. Then $\pi_1(F_t) = H_1(F_t)$ is abelian and for each map $p : F_s \rightarrow F_t$, $p_\pi : \pi_1(F_s) \rightarrow \pi_1(F_t)$ is a composition of $\sigma : \pi_1(F_s) \rightarrow H_1(F_s)$ and $p_\# : H_1(F_s) \rightarrow H_1(F_t)$, where σ is the abelianizing map, $p_\#$ is the map on homology. So the homotopy class of p is determined by $p_\#$ by elementary homotopy theory (see [3]).

Using this fact, we can construct γ_0 and γ_1 as following: choose essential curves α , β_0 and β_1 on F_s , see Figure 2, such that

- (1) β_0 and β_1 are in the same homology class, but not in the same homotopy class;
- (2) $|\alpha \cap \beta_0| = |\alpha \cap \beta_1| = 1$.

Let $\gamma_j = \partial N(\alpha \cup \beta_j)$, $V_j = N(\alpha \cup \beta_j)$. It is easy to check that $\gamma_0 \simeq \gamma_1$ and $p_{0\#} = p_{1\#} : H_1(F_s) \rightarrow H_1(F_t)$.

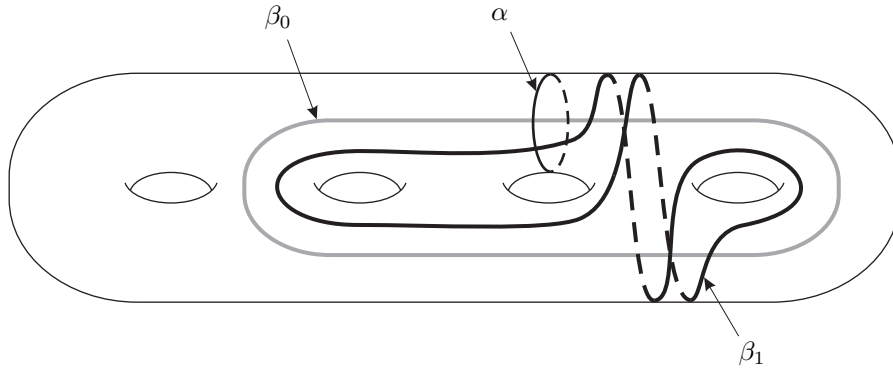


Figure 2

PROPOSITION 2.7. With the notation as above, once the two pinches $p_0, p_1 : F_s \rightarrow F_t$

are chosen to meet (i) and (ii) in Lemma 2.6, the extension f_s in Lemma 2.5 can be chosen to be pseudo-Anosov.

Suppose the two pinches $p_0, p_1 : F_s \rightarrow F_t$ are chosen to meet (i) and (ii) in Lemma 2.6, and $f_s : (F_s, V_0) \rightarrow (F_s, V_1)$ is an extension of $\bar{f}_t = e_1 \circ f_t| \circ e_0^{-1} : V_0 \rightarrow V_1$ with the condition $p_1 \circ f_s = f_t \circ p_0$.

LEMMA 2.8. (1) *If c is an essential circle in V , then c is essential in F_t .*
 (2) *No non-trivial circle $c \subset V_j$ can be isotoped into $W_{j'}$, $j, j' \in \{0, 1\}$, $j \neq j'$.*

Proof. (1) Otherwise c would bound a disk D^* in F_t with $\partial V \subset D^*$, hence c is parallel to ∂V in V , a contradiction.

(2) Otherwise say $c \subset V_0$ is a non-trivial circle, which is isotopic to a circle $c' \subset W_1$ in F_s .

First suppose that c is essential in V_0 . By $p_0 \sim p_1$ we have $p_0(c) \sim p_1(c')$. On one hand c is essential in V_0 implies that $p_0(c)$ is essential in V_0 , and then $p_0(c)$ is essential in F_t by (1). But on the other hand, $c' \subset W_1$ implies that $p_1(c')$ is homotopically trivial. We reach a contradiction.

Then suppose that ∂W_1 can be isotoped into W_0 . Then one of the two components V_1 and W_1 must be contained in W_0 . Since W_0 and W_1 are homeomorphic, if $W_1 \subset W_0$, we must have ∂W_0 is parallel to ∂W_1 , a contradiction. Hence $V_1 \subset W_0$, which implies that $\pi_1(V_1) \subset \ker p_{0\pi} = \ker p_{1\pi}$, which clearly is impossible.

So what remains to us is to modify $f_s|_{W_0}$.

Definition 2.9. [9] A set of mutually disjoint circles $\mathcal{C} = \{c_1, \dots, c_m\}$ on a compact surface F is an *independent set*, if the circles in \mathcal{C} are essential and mutually non-parallel.

LEMMA 2.10. *Let $h : W_1 \rightarrow W_1$ be a pseudo-Anosov map which is the identity in ∂W_1 . We extend h by identity to an automorphism h of F_s . \mathcal{A} is a maximal independent set of circles in W_0 . $f = f_s : F_s \rightarrow F_s$ is an extension of $\bar{f}_t : V_0 \rightarrow V_1$.*

Then when k is sufficiently large, for any $\alpha \in \mathcal{A}$, $h^k f(\alpha)$ is not isotopic to any circle in \mathcal{A} .

Proof. Suppose $k_1 < k_2$, $\alpha \in \mathcal{A}$. We claim that $h^{k_1} f(\alpha) \approx h^{k_2} f(\alpha)$ in F_s . In fact, $h^{k_1} f(\alpha)$ is an essential curve in W_1 , and any two curves in W_1 , which are homotopic in F_s , must be homotopic in W_1 . But $h|_{W_1}$ is a pseudo-Anosov automorphism on W_1 , so $h^{k_2-k_1}(h^{k_1} f(\alpha))$ is not isotopic to $h^{k_1} f(\alpha)$.

Hence for any $\alpha \in \mathcal{A}$, there are only finitely many k , such that $h^k f(\alpha)$ is homotopic to a circle in \mathcal{A} . Hence the conclusion holds.

From now on we replace f by $h^k f$.

Let \mathcal{A}_0 be a maximal independent set of circles on W_0 , \mathcal{A}_1 be its image under f_s . Let $V'_j = e_{s,j}(V_j)$, $W'_j = e_{s,j}(W_j)$, $\mathcal{A}'_j = e_{s,j}(\mathcal{A}_j)$. Let $L = o_s(\mathcal{A}' \cup \partial W'_0)$, $V' = o_s(V'_j)$, $W' = o_s(W'_j)$, $F' = o_s(F_s \times \{j\})$; and $X = M(F_s, f_s) - \text{int}(N(L))$, $F^* = F' \cap X$. Then

$$f'_s : (F_s \times \{0\}, V'_0, W'_0, \mathcal{A}_0) \rightarrow (F_s \times \{1\}, V'_1, W'_1, \mathcal{A}_1)$$

is a homeomorphism. We will prove that X is hyperbolic. We first have

LEMMA 2.11. (1) *\mathcal{A}'_j is a maximal independent set of circles on W'_j and each non-trivial circle in F^* is either essential in V'_j or parallel to a component of $\partial W'_j \cup \mathcal{A}'_j$.*

(2) *No component of $\partial W_0 \cup \mathcal{A}_0$ is homotopic to a component of $\partial W_1 \cup \mathcal{A}_1$, in F_s .*

Proof. (1) follows directly from the definitions and the constructions.

(2) ∂W_1 (resp. ∂W_0) can not be isotoped into W_0 (resp. W_1) by Lemma 2.8 (2). No component of $\mathcal{A}_1 = f_s(\mathcal{A}_0)$ is isotopic to a component of \mathcal{A}_0 by Lemma 2.10. Hence (2) follows.

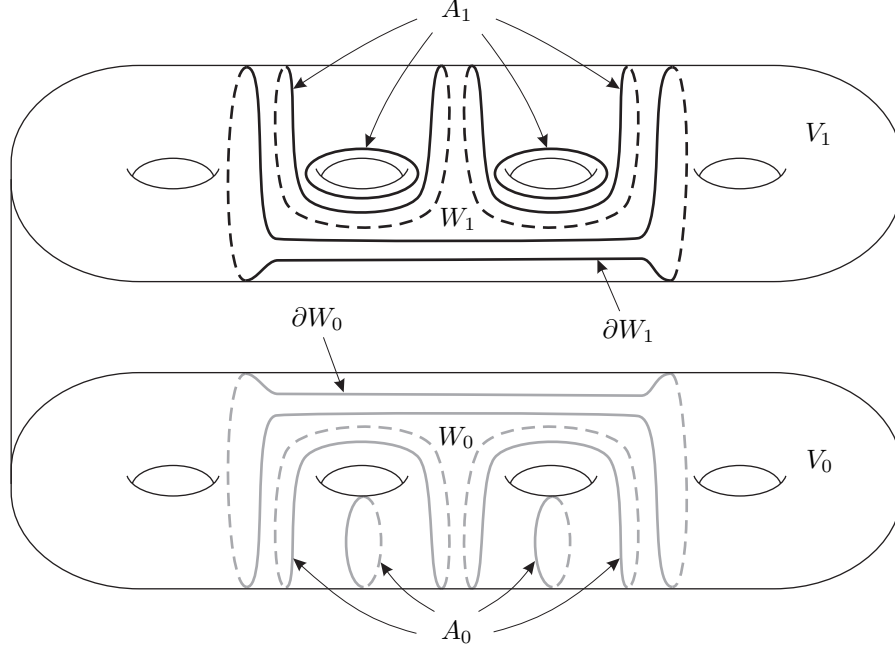


Figure 3

Figure 3 is what happens in $F_s \times [0, 1]$, and clearly illustrate the meaning of Lemma 2.11.

LEMMA 2.12. *X is atoroidal.*

Proof. Suppose that T is an essential torus in X . We assume that T has been isotoped in X so that $|T \cap F^*|$ is minimal. Then $T \cap F^* = \mathcal{C}^*$ consists of π_1 -injective circles on both T and F^* . Note $\mathcal{C}^* \neq \emptyset$, otherwise T would be an incompressible torus in $F_s \times [0, 1]$, which is impossible.

Cutting X along F^* , we get a manifold $X' \subset F_s \times [0, 1]$ and $T \setminus \mathcal{C}$ is a collection of annuli $A_1, A_2, \dots, A_n \subset X'$. In the case A_k is vertical, we denote the component of ∂A_k in $F_s \times \{j\}$ by $c_{k,j}$, and $q_s(c_{k,0}) = q_s(c_{k,1}) \subset F_s$ by c_k .

Now we claim that if one component of ∂A_k is essential in V'_j , then A_k is vertical. Because otherwise $\partial A_k \subset V'_j$, and we can push A_k across V' in X to reduce $|T \cap F^*|$, a contradiction.

By Lemma 2.11 (1), there are two cases:

Case 1. Some component of ∂A_k is an essential circle on V'_j .

By the claim above, A_k is vertical. Then $f'_s(c_{k,0})$ is a component of ∂A_l . ($l = k - 1$ or $k + 1$.) Since $f'_s(c_{k,0})$ is essential in V'_1 , it follows by the claim A_l is vertical and $f'_s(c_{k,0}) = c_{l,1}$. Then $f_s(c_k) = c_l$ and c_l is essential in V_1 . Hence c_l can not be isotoped into W_0 by Lemma 2.8 (2). Since $c_{l,0}$ is disjoint from $\partial W'_0$, $c_l \subset V_0$. Clearly c_l is still an

essential circle in V_0 , and therefore $c_{l,0}$ is essential in V'_0 . Since T is connected, repeat the same argument finitely many times, we get that

- (1) all A_k are vertical;
- (2) all $c_{k,0}$ are essential in V'_0 and all $c_{k,1}$ are essential in V'_1 ;
- (3) $f'_s(c_{k,0}) = c_{k+1,1}$ (re-indexing A_k if needed, and the subscript k is considered mod n). Hence each c_k is essential in both V_0 and V_1 , and $f_s(c_k) = c_{k+1}$.

Now both $p_0(c_k)$ and $p_1(c_k)$ are essential circles in V , and therefore essential in F_t by Lemma 2.8 (1). Since $f_t \circ p_0 = p_1 \circ f_s$, we have that $f_t \circ p_0(c_k) = p_1 \circ f_s(c_k) = p_1(c_{k+1})$. Since p_0 and p_1 are homotopic, we have $p_0(c_k) \sim p_1(c_k)$. Then up to isotopy $f_t^n \circ p_0(c_k) = p_0(c_k)$, which contradicts to the fact that f_t is a pseudo-Anosov map on F_t .

Case 2. Each component of ∂A_k is parallel to a component of $\partial W'_0 \cup \mathcal{A}'_0$. By Lemma 2.11 (2), no A_k is vertical, hence both components of ∂A_k are parallel to a component c of $\partial W'_0 \cup \mathcal{A}'_0$. So A_k can be rel ∂A_k isotoped into $N(c)$ in X' . Hence back to X the torus T can be isotoped into $N(o_s(c))$. This means that T is boundary parallel in X , contrary to our assumption.

It is easy to see that X is irreducible: a reducing sphere S would bound a ball B in $M(F_s, f_s)$, because $M(F_s, f_s)$, as a surface bundle over circle, is irreducible. Hence B would contain some component of L . This is impossible because each component of L is essential in $M(F_s, f_s)$.

X is not a Seifert fibered space: it contains $q(F_s \times \frac{1}{2})$, a non-separating, hyperbolic, closed incompressible surface. No such surface exists in a Seifert fibered space with boundary, because an essential surface in such a manifold is either horizontal (hence bounded) or vertical (hence a torus or an annulus).

Now the geometrization theorem of Thurston for Haken manifolds [8] leads us to the following:

COROLLARY 2.13. *X is a hyperbolic manifold.*

Suppose $L = \{\alpha_1, \dots, \alpha_m\}$, let T_l be the torus $\partial N(\alpha_l)$ on ∂X , $l = 1, \dots, m$. Denote by τ_c the right hand Dehn twist along a circle c on F_s . Pick a meridian-longitude pair for each T_l , with longitude a component of $F' \cap T_l$. q_l is a slope on T_l , define $X(q_1, \dots, q_m)$ to be the manifold obtained by q_l Dehn filling on T_l . The following lemma points a well-known relation between Dehn fillings and Dehn twists, which has been used in some papers, say [4] and [9].

LEMMA 2.14. *Let $\tilde{f}(k_1, \dots, k_m) = \tau_{\alpha_1}^{k_1} \circ \dots \circ \tau_{\alpha_m}^{k_m} \circ f$. Then*

$$M(F_s, \tilde{f}(k_1, \dots, k_m)) = X(1/k_1, \dots, 1/k_m)$$

for all $k_l \in \mathbb{Z}$.

Proof of Proposition 2.7. By Corollary 2.13, X is a hyperbolic manifold, therefore, by the hyperbolic surgery theorem of Thurston [7], $X(1/k_1, \dots, 1/k_m)$ is hyperbolic for sufficiently large k_l . The previous lemma implies that

$$X(1/k_1, \dots, 1/k_m) = M(F_s, \tilde{f}(k_1, \dots, k_m)).$$

The theorem now follows from Thurston's theorem that $M(F_s, \tilde{f})$ is hyperbolic if and only if \tilde{f} is isotopic to a pseudo-Anosov map ([8],[5]).

3. Adjusting Betti numbers

Now we pay attention to the Betti numbers of the surface bundles. Using HHN extension one can calculate directly that

$$H_1(M(F, f); \mathbb{Z}) = H_1(F, \mathbb{Z}) / \ker(l_{2g} - f_{\#}) \oplus \mathbb{Z}$$

where $g = g(F)$, l_{2g} is the unit matrix in $SL_{2g}(\mathbb{Z})$.

By abuse of notation, denote the image of $H_1(V_j)$ in $H_1(F_s)$ by $H_1(\widehat{V}_j)$. Similarly, define $H_1(\widehat{W}_j)$. We have

$$H_1(F_s) = H_1(\widehat{V}_0) \oplus H_1(\widehat{W}_0) = H_1(\widehat{V}_1) \oplus H_1(\widehat{W}_1).$$

LEMMA 3.1.

$$H_1(\widehat{V}_0) = H_1(\widehat{V}_1),$$

$$H_1(\widehat{W}_0) = H_1(\widehat{W}_1),$$

as subgroups of $H_1(F_s)$.

Proof. The second equation is easy, because $H_1(\widehat{W}_j) = \ker p_{j\#}$, and $p_0 \sim p_1$ implies $p_{0\#} = p_{1\#}$.

Now we will prove the first equation. Suppose $c_0 \subset V_0$ is a circle, then $c = p_0(c_0)$ is a circle in V , thus $c_1 = p_1^{-1}(c)$ is a circle in V_1 . Since $p_0 \sim p_1$, we have a map $H : F_s \times [0, 1] \rightarrow F_t$, $H(\cdot, j) = p_j(\cdot)$.

Make H transverse to c , then $H^{-1}(c)$ is a submanifold of $F_s \times [0, 1]$ with boundary $c_0 \cup c_1$. So c_0 and c_1 represent the same homology class in $H_1(F_s \times [0, 1]) = H_1(F_s)$. Hence the generators of $H_1(\widehat{V}_0)$ is the same as the ones of $H_1(\widehat{V}_1)$, our conclusion holds.

Choose a basis of $H_1(\widehat{V}_0)$ and a basis of $H_1(\widehat{W}_0)$ to make up a basis of $H_1(F_s)$. Under this basis, $f_{s\#}$ will be represented by a matrix of the form:

$$\begin{pmatrix} f_{t\#} & 0 \\ 0 & A \end{pmatrix}.$$

LEMMA 3.2. *The map f_s can be chosen so that the matrix $I - A$ is non-degenerate.*

Proof. Let $\delta = g_s - g_t$. Choose curves $\alpha_1, \dots, \alpha_\delta, \beta_1, \dots, \beta_\delta \subset W_1$, such that they are mutually disjoint, except that α_l intersects β_l in a single point transversely. These 2δ curves form a symplectic basis of $H_1(\widehat{W}_1)$. Under this basis, the intersection form of $H_1(\widehat{W}_1)$ is $\begin{pmatrix} 0 & I_\delta \\ -I_\delta & 0 \end{pmatrix}$. So if $f : (F_s, W_0) \rightarrow (F_s, W_1)$ is a homeomorphism, $f_{\#}|_{H_1(\widehat{W}_0)}$ will be represented by a symplectic matrix F .

We choose a map $\eta : W_1 \rightarrow W_1$, such that it fixes the points on ∂W_1 , and it induces F^{-1} on homology.

When $\delta > 1$, by Theorem 2 in [6], every symplectic matrix of rank 2δ can be represented by a pseudo-Anosov map on a closed surface of genus δ . So there is a pseudo-Anosov map $h : W_1 \rightarrow W_1$, such that it fixes the points on ∂W_1 , and induces $-l_{2\delta}$ on homology. Extend η, h to maps on F_s , with the points in V_1 fixed.

Let $\gamma_l = \partial N(\alpha_l \cup \beta_l)$, which is a separating circle in W_1 (see those separating circles in \mathcal{A}_1 , Figure 3). Now we extend $\{\alpha_1, \dots, \alpha_\delta, \gamma_1, \dots, \gamma_\delta\}$ to a maximal independent set \mathcal{A} on W_0 . Then every curve in $\mathcal{A} - \{\alpha_1, \dots, \alpha_\delta\}$ is homologous to 0. Let $L = \mathcal{A} \cup \{\partial W_1\}$. So the only Dehn twists along circles in L , which act nontrivially on homology group,

are $\tau_{\alpha_1}, \dots, \tau_{\alpha_\delta}$. The action of products of these twists on $H_1(\widehat{W}_1)$ is represented by a upper-triangular matrix T , whose diagonal elements are all 1. By Lemma 2.10, $h^{2k+1}\eta f$ does not send any curve in L into L when k is sufficiently large. The matrix of $h^{2k+1}\eta f$, when restricted on $H_1(\widehat{W}_0)$, is $-I_{2\delta}$. Now replace f by the composition of Dehn twists along L and $h^{2k+1}\eta f$, we have

$$I_{2\delta} - A = I_{2\delta} + T$$

is non-degenerate.

When $\delta = 1$, we give a direct construction. Let α, β be a symplectic basis of W_0 . Choose a map $h' : W_1 \rightarrow W_1$, with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $h'\eta f$ meets the conclusion of Lemma 2.10. τ is the Dehn twist along $\eta f(\alpha)$, then the matrix of τ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. τ_0 is the Dehn twist along ∂W_1 . Now the matrix A of $\tau_0^m \tau^k h'\eta f$ is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+k & 1+k \\ 1 & 1 \end{pmatrix}.$$

One can check $I - A$ is non-degenerate when $k \geq 0$.

To prove Theorem 1.3 (2), we need only to show the following lemma.

LEMMA 3.3. *With the notation as above, the extension f_s in Proposition 2.7 can be chosen so that*

$$\text{rank}(H_1(F_s; \mathbb{Q}) / \ker(I_{2g_s} - f_{s\#})) = \text{rank}(H_1(F_t, \mathbb{Q}) / \ker(I_{2g_t} - f_{t\#})).$$

Proof. The conclusion follows from the formula of computing $H_1(M(F, f))$ and Lemma 3.2.

Proof of Theorem 1.3. This theorem follows from Proposition 2.7 and Lemma 3.3.

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